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Hidden Conformal Symmetry of Extremal Black Holes

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Abstract: We study the hidden conformal symmetry of extremal black holes. We introduce a new set of conformal coordinates to write the $SL(2,R)$ generators. We find that the Laplacian of the scalar field in many extremal black holes, including Kerr(-Newman),RN, warped AdS$_3$ and null warped black holes, could be written in terms of the $SL(2,R)$ quadratic Casimir. This suggests that there exist dual conformal field theory (CFT) descriptions of these black holes. From the conformal coordinates, the temperatures of the dual CFTs could be read directly. For the extremal black hole, the Hawking temperature is vanishing. Correspondingly, only the left (right) temperature of the dual CFT is nonvanishing and the excitations of the other sector are suppressed. In the probe limit, we compute the scattering amplitudes of the scalar off the extremal black holes and find perfect agreement with the CFT prediction.

Contents

1. Introduction

The AdS/CFT correspondence[1] states that the quantum gravity in anti-de-Sitter(AdS) spacetime is dual to a conformal field theory(CFT) at the AdS boundary. This discovery opens a new window to study the quantum gravity. In this correspondence, the black hole asymptotic to the AdS spacetime could be described by the CFT at a finite temperature. In [2], it was conjectured that even asymptotically flat four-dimensional Kerr black hole may have a holographic two-dimensional CFT description. This conjecture was first set up for extremal and near-extremal Kerr black hole by studying their near-horizon geometry[2, 3, 4, 5] and super-radiant scattering[6, 7, 8, 9, 10]. Very recently, this Kerr/CFT correspondence was generalized to the generic nonextremal Kerr black hole[11]. The key ingredient is the hidden conformal symmetry of the Kerr black hole.

For a generic nonextremal Kerr black hole, the hidden conformal symmetry is realized by introducing a set of conformal coordinates:

$$\begin{aligned}
\omega^+ &= \sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi T_R\phi+2n_R t}, \\
\omega^- &= \sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi T_L\phi+2n_L t}, \\
y &= \sqrt{\frac{r_+-r_-}{r-r_-}}e^{\pi(T_L+T_R)\phi+(n_L+n_R)t},
\end{aligned} \tag{1.1}$$

where $r_\pm$ are the horizons of the black hole. With $(\omega^\pm, y)$, one can locally define two sets of vector fields satisfying the $SL(2,R)$ Lie algebra. More importantly, the scalar Laplacian in the low-frequency limit could be written as the $SL(2,R)$ quadratic Casimir, showing hidden $SL(2,R)\times SL(2,R)$ symmetries. Even though these $SL(2,R)$ symmetries are not globally defined and are broken by the angular identification $\phi \sim \phi + 2\pi$, they act on the solution space, and determine the form of the scattering amplitudes. It is also remarkable that the conformal coordinates $\omega^\pm$ encode the information of the temperatures of the dual CFT, as a result of the Unruh effect. The existence of this hidden conformal symmetry is essential to set up a CFT dual to nonextremal Kerr black hole. The study of hidden conformal symmetry has been generalized to other kinds of black hole and applied to compute the real-time correlators in Kerr/CFT, see [12]-[26].

However, the conformal coordinates (1.1) do not make sense for extremal black holes. In this case, as $r_+ = r_-$ the coordinate y is simply zero and not well-defined. On the other hand, it could be expected that the hidden conformal symmetry should be still there. In this paper, we try to address this issue. We introduce a new set of conformal coordinates and compute the corresponding $SL(2,R)$ quadratic Casimir. We find that for many extremal black hole, the scalar radial equation could be written as the $SL(2,R)$ Casimir, showing that the hidden conformal symmetries for extremal black holes do exist. We read the dual temperature from the conformal coordinate directly and find agreement with the known one in the extremal limit. We furthermore discuss the scattering amplitudes off the extremal black hole and find good agreement with the CFT prediction.

In the next section, we introduce the new conformal coordinates and discuss the hidden conformal symmetry. In Sec. 3, we study the extremal Kerr(-Newman) black hole. In Sec. 4, we turn to other extremal black holes, including warped AdS$_3$, null warped, the near-horizon geometry of extreme Kerr black hole (NHEK) and 4D RN black hole. We end with some conclusions in Sec. 5.

2. Hidden conformal symmetry

For extreme black holes, the conformal coordinates are very different from the ones of nonextremal black holes widely used in the literature. They could be defined as follows

$$\begin{aligned}
\omega^+ &= \frac{1}{2}\left(\alpha_1 t + \beta_1 \phi - \frac{\gamma_1}{r - r_+}\right), \\
\omega^- &= \frac{1}{2}\left(e^{2\pi T_L \phi + 2n_L t} - \frac{2}{\gamma_1}\right), \\
y &= \sqrt{\frac{\gamma_1}{2(r - r_+)}}\, e^{\pi T_L \phi + n_L t}.
\end{aligned} \tag{2.1}$$

With them, we can locally define the vector fields

$$\begin{aligned}
H_1 &= i\partial_+ \\
H_0 &= i\left(\omega^+\partial_+ + \frac{1}{2} y \partial_y\right) \\
H_{-1} &= i(\omega^{+2}\partial_+ + \omega^+ y \partial_y - y^2 \partial_-)
\end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
\tilde{H}_1 &= i\partial_- \\
\tilde{H}_0 &= i\left(\omega^- \partial_- + \frac{1}{2} y\partial_y\right) \\
\tilde{H}_{-1} &= i(\omega^{-2}\partial_- + \omega^- y\partial_y - y^2\partial_+)
\end{aligned} \tag{2.3}
$$

These vector fields obey the $SL(2,R)$ Lie algebra

$$
[H_0, H_{\pm 1}] = \mp i H_{\pm 1}, \qquad [H_{-1}, H_1] = -2iH_0, \tag{2.4}
$$

and similarly for $(\tilde{H}_0, \tilde{H}_{\pm 1})$. The quadratic Casimir is

$$
\begin{aligned}
\mathcal{H}^2 = \tilde{\mathcal{H}}^2 &= -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1} H_1) \\
&= \frac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2\partial_+\partial_- .
\end{aligned} \tag{2.5}
$$

In terms of (t, r, ϕ) coordinates, the vector fields are of the form:

$$
\begin{aligned}
H_1 &= i\frac{2}{A}(2\pi T_L \partial_t - 2n_L\partial_\phi) \\
H_0 &= i\left(-(r-r_+)\partial_r + \frac{1}{A}(\alpha_1 t + \beta_1\phi)(2\pi T_L\partial_t - 2n_L\partial_\phi)\right) \\
H_{-1} &= i\left\{-(\alpha_1 t + \beta_1\phi)(r-r_+)\partial_r + \frac{\gamma_1}{A(r-r_+)}(\beta_1\partial_t - \alpha_1\partial_\phi)\right. \\
&\quad \left. + \left((\alpha_1 t + \beta_1\phi)^2 + \frac{\gamma_1^2}{(r-r_+)^2}\right)\frac{1}{2A}(2\pi T_L\partial_t - 2n_L\partial_\phi)\right\} \\
\tilde{H}_1 &= 2ie^{-2\pi T_L\phi - 2n_L t}\left((r-r_+)\partial_r - \frac{1}{A}(\beta_1\partial_t - \alpha_1\partial_\phi) - \frac{\gamma_1}{A(r-r_+)}(2\pi T_L\partial_t - 2n_L\partial_\phi)\right) \\
\tilde{H}_0 &= i\left(-\frac{2}{\gamma_1}e^{-2\pi T_L\phi - 2n_L t}(r-r_+)\partial_r - (1 - \frac{2}{\gamma_1}e^{-2\pi T_L\phi - 2n_L t})\frac{1}{A}(\beta_1\partial_t - \alpha_1\partial_\phi)\right. \\
&\quad \left. + \frac{2e^{-2\pi T_L\phi - 2n_L t}}{A(r-r_+)}(2\pi T_L\partial_t - 2n_L\partial_\phi)\right) \\
\tilde{H}_{-1} &= i\left\{-\frac{1}{2}\left(e^{2\pi T_L\phi + 2n_L t} - \frac{4}{\gamma_1^2}e^{-2\pi T_L\phi - 2n_L t}\right)(r-r_+)\partial_r\right. \\
&\quad - \left(e^{2\pi T_L\phi + 2n_L t} - \frac{4}{\gamma_1} + \frac{4}{\gamma_1^2}e^{-2\pi T_L\phi - 2n_L t}\right)\frac{1}{2A}(\beta_1\partial_t - \alpha_1\partial_\phi) \\
&\quad \left. - \left(e^{2\pi T_L\phi + 2n_L t} + \frac{4}{\gamma_1^2}e^{-2\pi T_L\phi - 2n_L t}\right)\frac{\gamma_1}{2A(r-r_+)}(2\pi T_L\partial_t - 2n_L\partial_\phi)\right\}
\end{aligned}
$$

and the quadratic Casimir becomes

$$
\mathcal{H}^2 = \partial_r(\Delta\partial_r) - \left(\frac{\gamma_1(2\pi T_L\partial_t - 2n_L\partial_\phi)}{A(r-r_+)}\right)^2 - \frac{2\gamma_1(2\pi T_L\partial_t - 2n_L\partial_\phi)}{(r-r_+)A^2}(\beta_1\partial_t - \alpha_1\partial_\phi), \tag{2.6}
$$

where $A = 2\pi T_L\alpha_1 - 2n_L\beta_1$ and $\Delta = (r-r_+)^2$.

Actually, there exist some degrees of freedom to define the conformal coordinates (2.1) without affect the form of the Casimir. For example, the γ_1 term in ω^- is redundant and could be abandoned and there could be a free overall scale factor in ω^-. However, such degrees of freedom do not change the underlying physics.

Similar to the hidden conformal symmetry of the nonextremal Kerr-Newman black hole, the vector fields are not globally defined. The periodic identification along ϕ breaks the conformal symmetry. As argued in [11], from the relation between ω^- and $t^- = 2\pi T_L + 2n_L t$, one may take t^- as the Rindler coordinate such that the observer in it will observe a thermal bath of Unruh radiation with temperature T_L. Note that for extremal black holes, only one sector in the dual CFT is excited.

It turns out that the radial equations of the extremal black holes always take the form

$$\mathcal{H}^2\Phi(r) = K\Phi(r), \tag{2.7}$$

where K is a r-independent parameter contributing the conformal weight. With the ansatz $\Phi(r) = e^{-i\omega t+im\phi}R(r)$, the radial equation can be written as

$$\partial_r\Delta\partial_r R(r) + \frac{\mu^2}{(r-r_+)^2}R(r) + \frac{\nu}{r-r_+}R(r) = KR(r), \tag{2.8}$$

where

$$\begin{aligned}\mu &= \frac{\gamma_1(2\pi T_L\omega + 2n_L m)}{A}\\ \nu &= \frac{2\gamma_1(2\pi T_L\omega + 2n_L m)(\beta_1\omega + \alpha_1 m)}{A^2}.\end{aligned} \tag{2.9}$$

Introducing $z = \frac{-2i\mu}{r-r_+}$, we get the Whittaker equation

$$R''(z) + (-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m_s^2}{z^2})R(z) = 0, \tag{2.10}$$

where

$$k = \frac{i(\beta_1\omega + \alpha_1 m)}{A}, \qquad m_s^2 = \frac{1}{4} + K. \tag{2.11}$$

This equation has the solution

$$R(z) = C_1R_+(z) + C_2R_-(z), \tag{2.12}$$

where $R_\pm(z) = e^{-\frac{z}{2}}z^{\frac{1}{2}\pm m_s}F(\frac{1}{2}\pm m_s - k, 1\pm 2m_s, z)$ are two linearly independent solution. Near the horizon $r \to r_+$ so $z \to \infty$, the Kummer function could be expanded asymptotically

$$F(\alpha,\gamma,z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}e^{-i\alpha\pi}z^{-\alpha} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^z z^{\alpha-\gamma}. \tag{2.13}$$

As we need to impose purely ingoing boundary condition at the horizon, we have to require

$$C_1 = -\frac{\Gamma(1-2m_s)}{\Gamma(\frac{1}{2}-m_s-k)}C, \qquad C_2 = \frac{\Gamma(1+2m_s)}{\Gamma(\frac{1}{2}+m_s-k)}C \tag{2.14}$$

to cancel the outgoing modes, where C is a constant.

When r goes asymptotically to infinity, $z \to 0$, $F(\alpha, \gamma, z) \to 1$, the solution has asymptotical behavior

$$R \sim C_1 r^{-h} + C_2 r^{1-h}, \tag{2.15}$$

where h is the conformal weight of the scalar

$$h = \frac{1}{2} + m_s = \frac{1}{2} + \sqrt{\frac{1}{4} + K}. \tag{2.16}$$

The retarded Green's function could be read directly[27]

$$G_R \sim \frac{C_1}{C_2} \propto \frac{\Gamma(1-2h)\Gamma(h-k)}{\Gamma(2h-1)\Gamma(1-h-k)}. \tag{2.17}$$

3. Extremal Kerr-Newman black hole

For the Kerr-Newman black hole with mass M, angular momentum $J = aM$ and electric charge Q, its metric takes the following form

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{1}{\rho^2}\sin^2\theta\left(adt - (r^2+a^2)d\phi\right)^2, \tag{3.1}$$

where

$$\begin{aligned} \Delta &= (r^2 + a^2) - 2Mr + Q^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta. \end{aligned} \tag{3.2}$$

The gauge field is

$$A = -\frac{Qr}{\rho^2}(dt - a\sin^2\theta d\phi). \tag{3.3}$$

There are two horizons located at

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{3.4}$$

And the Hawking temperature, entropy, angular velocity of the horizon and the electric potential are respectively

$$\begin{aligned} T_H &= \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}, \\ S &= \pi(r_+^2 + a^2), \\ \Omega_H &= \frac{a}{r_+^2 + a^2}, \\ \Phi &= \frac{Qr_+}{r_+^2 + a^2}. \end{aligned} \tag{3.5}$$

Let us consider the complex scalar field with mass μ and charge e scattering with the Kerr-Newman black hole. The Klein-Gordon equation is

$$(\nabla_\mu + ieA_\mu)(\nabla^\mu + ieA^\mu)\Phi - \mu^2\Phi = 0. \tag{3.6}$$

With the ansatz

$$\Phi = e^{-i\omega t + im\phi}\mathcal{R}(r)\mathcal{S}(\theta), \tag{3.7}$$

where ω and m are the quantum numbers, the wave equation could be decomposed into the angular part and the radial part. The angular part is of the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\mathcal{S}\right) + \left(\Lambda_{lm} - a^2(\omega^2-\mu^2)\sin^2\theta - \frac{m^2}{\sin^2\theta}\right)\mathcal{S} = 0. \tag{3.8}$$

Here Λ_{lm} is the separation constant. It is restricted by the regularity boundary condition at $\theta = 0, \pi$ and can be computed numerically. The radial part of the wave function is of the form

$$\partial_r(\Delta\partial_r\mathcal{R}) + V_R\mathcal{R} = 0 \tag{3.9}$$

with

$$V_R = -\Lambda_{lm} + 2am\omega + \frac{H^2}{\Delta} - \mu^2(r^2+a^2), \tag{3.10}$$

$$H = \omega(r^2+a^2) - eQr - am. \tag{3.11}$$

As we are interested in the low-frequency limit,

$$\omega M << 1, \tag{3.12}$$

the ω^2 term in the angular equation could be neglected. Note that the low-frequency limit (3.12) is very different from the case studied in [6, 7], where only the frequencies near the superradiant bound were studied. To simplify our discussion, we focus on the massless scalar. Then the angular equation is just the Laplacian on the 2-sphere with the separation constants taking values

$$\Lambda_{lm} = l(l+1). \tag{3.13}$$

In the near region, $r\omega << 1$, the radial equation could be simplified even more

$$\begin{aligned}\partial_r\Delta\partial_r\mathcal{R}(r) &+ \frac{(ma - \omega(2Mr_+ - Q^2) + eQr_+)^2}{(r-r_+)(r_+-r_-)}\mathcal{R}(r) \\ &- \frac{(ma - \omega(2Mr_- - Q^2) + eQr_-)^2}{(r_+-r_-)(r-r_-)}\mathcal{R}(r) = \left(l(l+1) - e^2Q^2\right)\mathcal{R}(r)\end{aligned} \tag{3.14}$$

Let us consider the extreme Kerr-Newman black hole. In this case, the Hawking temperature is vanishing but the entropy and the angular velocity is still nonzero. In the low-frequency limit, we have the radial equation as

$$\begin{aligned}\partial_r\Delta\partial_r\mathcal{R}(r) &+ \frac{2(2M\omega - eQ)((2Mr_+ - Q^2)\omega - am - eQr_+)}{(r-r_+)}\mathcal{R}(r) \\ &+ \frac{((2Mr_+ - Q^2)\omega - am - eQr_+)^2}{(r-r_+)^2}\mathcal{R}(r) = \left(l(l+1) - e^2Q^2\right)\mathcal{R}(r).\end{aligned} \tag{3.15}$$

For the neutral scalar with $e = 0$, it is easy to see that the left hand side of this equation could be rewritten as the $SL(2,R)$ Casimir (2.6) with the identification

$$\alpha_1 = 0, \quad \beta_1 = \gamma_1/a, \quad 2\pi T_L = \frac{2r_+ - Q^2/M}{2a}, \quad n_L = -\frac{1}{4M}. \tag{3.16}$$

The identifications of T_L and n_L are consist with the existing result[15, 14].

3.1 Microscopic description

Now the microscopic entropy comes from only the left sector

$$S = \frac{\pi^2}{3} c_L T_L = \pi(r_+^2 + a^2), \tag{3.17}$$

in agreement with the macroscopic Bekenstein-Hawking entropy.

We can still determine the conjugate charge from the first law of thermodynamics. In the extreme case, the Hawking temperature $T_H = 0$, we should consider it carefully. We begin with a nonzero T_H, and then take limit to set it to 0. When $T_H \neq 0$,

$$M^2 - a^2 - Q^2 > 0. \tag{3.18}$$

From the first law of thermodynamics,

$$\delta S = \frac{\delta M - \Omega_H \delta J - \Phi \delta Q}{T_H} \tag{3.19}$$

we get

$$\delta S = 2\pi(2M\delta M - Q\delta Q) + 4\pi \frac{(2M^2 - Q^2)\delta M - a\delta J - QM\delta Q}{2\sqrt{M^2 - a^2 - Q^2}} \tag{3.20}$$

To analyze the second term, we introduce

$$a = M(1-\epsilon)\cos\theta, \qquad Q = M(1-\epsilon)\sin\theta \tag{3.21}$$

where ϵ and θ are two parameter to control the $T_H \to 0$ limit. Note that only ϵ is really related to the limit $T_H \to 0$, so we may change M and θ simultaneously:

$$\begin{aligned} \delta J &= 2M(1-\epsilon)\cos\theta\delta M - M^2(1-\epsilon)\sin\theta\delta\theta, \\ \delta Q &= (1-\epsilon)\sin\theta\delta M + M(1-\epsilon)\cos\theta\delta\theta, \end{aligned}$$

then

$$\frac{(2M^2 - Q^2)\delta M - a\delta J - QM\delta Q}{2\sqrt{M^2 - a^2 - Q^2}} = M\delta M\sqrt{1-(1-\epsilon)^2}. \tag{3.22}$$

In the limit $T_H \to 0$ the second term turns to zero.

Next, we consider the first term. We find that if we identify

$$\delta Q = e, \qquad \delta M = \omega, \qquad \delta E_L = \omega_L - q_L\mu_L, \tag{3.23}$$

with

$$\omega_L = \frac{(2M^2 - Q^2)M}{J}\omega, \qquad \mu_L = \frac{QM^2}{J} - \frac{Q^3}{2J}, \qquad q_L = e, \tag{3.24}$$

then

$$\delta S = \frac{\delta E_L}{T_L} = \frac{\omega_L - q_L\mu_L}{T_L}. \tag{3.25}$$

Note that the identification (3.24) is the same as the one found in the study of the nonextremal Kerr-Newmann black hole. The above discussion also suggests that only the left

mover in the dual CFT is relevant to the extremal black hole, while the right mover is kept completely silent.

The retarded charged scalar Green's function in the extremal Kerr-Newman black hole could be rewritten as

$$\begin{aligned} G_R &\sim \frac{\Gamma(1-2h)\Gamma(h-i(2M\omega-eQ))}{\Gamma(2h-1)\Gamma(1-h-i(2M\omega-eQ))} \\ &= \frac{\Gamma(1-2h)}{\Gamma(2h-1)}\frac{\Gamma\left(h-i\frac{\omega_L-q_L\mu_L}{2\pi T_L}\right)}{\Gamma\left(1-h-i\frac{\omega_L-q_L\mu_L}{2\pi T_L}\right)}. \end{aligned} \tag{3.26}$$

Now the conformal weight $h=\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)-e^2Q^2}$.

In a two-dimensional conformal field theory, the two-point functions of the primary operators are determined by the conformal invariance[28]. The retarded correlator $G_R(\omega_L,\omega_R)$ is analytic on the upper half complex $\omega_{L,R}$ plane and its value along the positive imaginary $\omega_{L,R}$ axis gives the Euclidean correlator:

$$G_E(\omega_{L,E},\omega_{R,E}) = G_R(i\omega_{L,E}, i\omega_{R,E}), \quad \omega_{L,E},\omega_{R,E}>0. \tag{3.27}$$

At finite temperature, $\omega_{L,E}$ and $\omega_{R,E}$ take discrete values of the Matsubara frequencies

$$\omega_{L,E}=2\pi m_L T_L, \quad \omega_{R,E}=2\pi m_R T_R, \tag{3.28}$$

where m_L, m_R are integers for bosonic modes and are half integers for fermionic modes. For an operator of dimensions (h_L,h_R), charges (q_L,q_R) at temperatures (T_L,T_R) and chemical potentials (μ_L,μ_R), the momentum space Euclidean correlator is given by[29]

$$\begin{aligned} G_E(\omega_{L,E},\omega_{R,E}) \sim\ & T_L^{2h_L-1}T_R^{2h_R-1}e^{i\frac{\bar\omega_{L,E}}{2T_L}}e^{i\frac{\bar\omega_{R,E}}{2T_R}} \\ & \cdot\Gamma(h_L+\frac{\bar\omega_{L,E}}{2\pi T_L})\Gamma(h_L-\frac{\bar\omega_{L,E}}{2\pi T_L})\Gamma(h_R+\frac{\bar\omega_{R,E}}{2\pi T_R})\Gamma(h_R-\frac{\bar\omega_{R,E}}{2\pi T_R}), \end{aligned} \tag{3.29}$$

where

$$\bar\omega_{L,E}=\omega_{L,E}-iq_L\mu_L, \quad \bar\omega_{R,E}=\omega_{R,E}-iq_R\mu_R. \tag{3.30}$$

In our case, only the left part of the CFT is necessary, then

$$G_E(\omega_{L,E}) \sim T_L^{2h_L-1}e^{i\frac{\bar\omega_{L,E}}{2T_L}}\Gamma(h_L+\frac{\bar\omega_{L,E}}{2\pi T_L})\Gamma(h_L-\frac{\bar\omega_{L,E}}{2\pi T_L}). \tag{3.31}$$

The real-time correlator (3.26) is obviously in agreement with the above relation.

3.2 Kerr case

When $Q=0$, the Kerr-Newman black hole reduces to the Kerr black hole. For the extreme Kerr, we have

$$a=M, \quad T_L=\frac{1}{2\pi}, \quad \omega_L=2M\omega. \tag{3.32}$$

In this case, we can study the scattering amplitudes for general spin s[7, 15]. The radial equation is

$$\Delta^{-s}\partial_r\Delta^{s+1}\partial_r R(r)+[\frac{H^2-2is(r-r_+)}{\Delta}+4is\omega r+2ma\omega+s(s+1)-K]R(r)=0 \quad (3.33)$$

where $H=\omega(r^2+a^2)-ma$ and K is the separation constant. When we consider the low-frequency limit, we get the following equation

$$\Delta^{-s}\partial_r\Delta^{s+1}\partial_r R(r)+(s(s+1)-K)R(r)+\frac{d}{r-r_+}R(r)+\frac{e^2}{(r-r_+)^2}R(r)=0 \quad (3.34)$$

where

$$\begin{aligned} d &= (4M\omega-2is)(2M\omega r_+-ma), \\ e &= 2M\omega r_+-ma. \end{aligned}$$

Setting $R(r)=(r-r_+)^{-s}\psi(r)$, we get the following equation

$$\partial_r\Delta\partial_r\psi+\frac{d}{r-r_+}\psi+\frac{e^2}{(r-r_+)^2}\psi=K\psi, \quad (3.35)$$

which is of the same as the Eq. (2.8). The retarded Green's function could be obtained in the way suggested in [9]. The result is

$$G\sim\frac{\Gamma(1-2h)\Gamma(h_L-i\frac{\omega_L}{2\pi T_L})}{\Gamma(2h-1)\Gamma(1-h_L-i\frac{\omega_L}{2\pi T_L})} \quad (3.36)$$

where $h_L=h-s$. This is in agreement with the CFT prediction.

4. Other extreme black holes

There has been much discussion on the hidden conformal symmetry acting on the solution space of the scalar field in other black holes. We will show that for the extremal cases, there exist hidden conformal symmetry as well. In all the cases we study, the conformal coordinates proposed in Sec. 2 work well.

4.1 Extreme warped AdS$_3$ black hole

The spacelike stretched AdS_3 spacetime is the vacuum solution of three-dimensional topological massive gravity[30, 31]. The spacelike stretched AdS_3 black hole could be constructed from discrete identification along a Killing vector of the global warped spacetime, similar to the BTZ black hole[32].

The metric of the spacelike stretched warped AdS_3 black hole takes the following form in terms of Schwarzschild coordinates:

$$ds^2=l^2(\mathrm{d}t^2+2M(r)\mathrm{d}t\mathrm{d}\theta+N(r)\mathrm{d}\theta^2+D(r)\mathrm{d}r^2), \quad (4.1)$$

where

$$
\begin{aligned}
M(r) &= vr - \frac{1}{2}\sqrt{r_+ r_-(v^2+3)}, \\
N(r) &= \frac{r}{4}\left(3(v^2-1)r + (v^2+3)(r_+ + r_-) - 4v\sqrt{r_+ r_-(v^2+3)}\right), \\
D(r) &= \frac{1}{(v^2+3)(r-r_+)(r-r_-)},
\end{aligned}
$$

and $-l^{-2}$ is the negative cosmological constant and the parameter $v = \mu l/3$ with μ being the mass of the graviton. There are two horizons located at $r = r_+$ and $r = r_-$. It is only well behaved for $\nu > 1$. The hidden conformal symmetry in this black hole has been discussed in [22].

From the warped AdS/CFT correspondence[33], this black hole could be described by a two-dimensional conformal field theory with temperatures

$$
T_L = \frac{(v^2+3)}{8\pi l}\left(r_+ + r_- - \frac{\sqrt{(v^2+3)r_+ r_-}}{v}\right), \tag{4.2}
$$

$$
T_R = \frac{(v^2+3)(r_+ - r_-)}{8\pi l}, \tag{4.3}
$$

and central charges

$$
c_L = \frac{l}{G}\frac{4v}{v^2+3}, \qquad c_R = \frac{l}{G}\frac{5v^2+3}{v(v^2+3)}. \tag{4.4}
$$

The scalar field of mass μ in this background obeys the equation of motion:

$$
(\nabla_\nu \nabla^\nu - \mu^2)\Phi = 0. \tag{4.5}
$$

Since the background has the translational isometry along t and θ, we may make the following ansatz

$$
\Phi = e^{-i\omega t + ik\theta}\phi. \tag{4.6}
$$

For the extremal warped black hole, the scalar equation turns to be

$$
\begin{aligned}
&\partial_r \Delta \partial_r \phi(r) + \frac{4v\omega((2v-\sqrt{v^2+3})r_+\omega + 2k)}{(r-r_+)(v^2+3)^2}\phi(r) \\
&+ \frac{((2v-\sqrt{v^2+3})r_+\omega + 2k)^2}{(r-r_+)^2(v^2+3)^2}\phi(r) = \left(\frac{\mu^2 l^2}{v^2+3} - \frac{3(v^2-1)}{(v^2+3)^2}\omega^2\right)\phi(r)
\end{aligned} \tag{4.7}
$$

Note that the right hand side is closely related to the conformal weight of the scalar field, taken into account of the quantum number identification. For the identification of quantum numbers, please see [35, 36] for details. One does not need to take the low-frequency limit as done in [22], which makes the warped AdS/CFT correspondence less clear. Moreover, the hidden conformal symmetry exist in the whole spacetime, rather than just the "Near" region in the Kerr case. The left hand side of the equation could be written as the $SL(2,R)$ quadratic Casimir (2.6) with the identifications

$$
\begin{aligned}
&\alpha_1 = 0, \quad \beta_1 = \frac{v^2+3}{2}\gamma_1 \\
&2\pi T_L = \frac{v^2+3}{4vl}(2v - \sqrt{v^2+3})r_+, \quad n_L = \frac{v^2+3}{4vl}.
\end{aligned} \tag{4.8}
$$

The left temperature is exactly the same as the one in (4.2). Similar to the Kerr/CFT case, for the extreme black holes, only the left temperature is nonvanishing. The scattering amplitude could be discussed straightforwardly, taking care of the subtle identification of quantum numbers found in [34, 35]. The result is in perfect with the CFT prediction.

4.2 Null warped black holes

The null warped AdS_3 spacetime is another vacuum solution of three-dimensional topological massive gravity. It is only well defined at $v = 1$. The null warped black hole could be taken as the quotient of the null warped AdS_3. The metric of the null warped black hole is of the form

$$\frac{ds^2}{l^2} = -2r\mathrm{d}\theta\mathrm{d}t + (r^2 + r + \alpha^2)\mathrm{d}\theta^2 + \frac{\mathrm{d}r^2}{4r^2}, \tag{4.9}$$

where $1/2 > \alpha > 0$ in order to avoid the naked causal singularity. The horizon is located at $r = 0$. Different from its spacelike stretched cousin, this black hole is extreme by construction. From the null warped AdS/CFT correspondence[33, 35], it was suggested to be described by a dual two-dimensional CFT with only nonvanishing right-moving temperature

$$T_R = \frac{\alpha}{\pi l} \tag{4.10}$$

and the central charge[37]

$$C_R = 2l/G. \tag{4.11}$$

In this case, there is only right central charge in the dual CFT, suggesting that the dual CFT is chiral.

The equation of motion for the scalar field of mass μ is

$$\nabla^2\Phi - \mu^2\Phi = 0. \tag{4.12}$$

Taken the ansatz $\Phi = e^{-i\omega t + ik\theta}R(r)$, the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2\frac{\mathrm{d}}{\mathrm{d}r}R) + \left(\frac{\omega^2 - 2\omega k}{4r} + \frac{\omega^2\alpha^2}{4r^2}\right) R = \frac{\mu^2 l^2 - \omega^2}{4}R. \tag{4.13}$$

Similar to the spacelike warped black hole, the right hand side of the equation is related to the conformal weight of the scalar field. The left hand side of the equation could be written as the quadratic Casimir (2.6) with the following identification

$$\alpha_1 = -2\beta_1, \quad \gamma_1 = -\frac{\alpha}{2}\alpha_1, \quad n_L = 0, \quad T_L = \frac{\alpha}{\pi l}. \tag{4.14}$$

The scattering amplitudes of null warped black hole have been discussed in detail in [36].

4.3 NHEK

The near-horizon geometry of extreme Kerr black hole (NHEK) was first discovered in [5]. It played a key role in setting up the Kerr/CFT correspondence. The NHEK geometry in Poincare-type coordinates is

$$ds^2 = \Gamma\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \Lambda^2(d\phi + rdt)^2\right). \tag{4.15}$$

where

$$\Gamma(\theta) = \frac{1+\cos^2\theta}{2}, \qquad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}. \tag{4.16}$$

It was shown in [9] that the NHEK geometry could be taken as the extreme black hole, with the horizon located at $r = 0$.

With the ansatz

$$\Phi = e^{-i\omega t + im\phi}\mathcal{R}(r)\mathcal{S}(\theta), \tag{4.17}$$

the radial function obeys

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\mathcal{R}(r) + \left(\frac{\omega^2}{r^2} + \frac{2\omega m}{r}\right)\mathcal{R}(r) = \left(\Lambda_{lm} - 2m^2\right)\mathcal{R}(r). \tag{4.18}$$

The coefficient in the right hand side is related to the conformal weight. The left hand side can be recast into the quadratic Casimir (2.6) with the following identification

$$\beta_1 = 0, \qquad \gamma_1 = \alpha_1 \qquad n_L = 0, \qquad T_L = \frac{1}{2\pi}. \tag{4.19}$$

The temperature is exactly the same as the one found in [2]. The scattering amplitudes for various kinds of perturbations in NHEK have been discussed in [9].

A similar discussion could easily be applied to the extremal self-dual warped black hole[38].

4.4 Extreme RN black hole

The four-dimensional electrically charged RN black hole is described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{4.20}$$

with

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{4.21}$$

and the gauge potential

$$A = -\frac{2Q}{r}dt. \tag{4.22}$$

The Hawking temperature and entropy are respectively

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2}, \qquad S = \pi r_+^2, \tag{4.23}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. In order to study the CFT dual of the four-dimensional RN black hole, one has to embed it into five dimension. From the RN/CFT correspondence[39], it is described by a dual CFT with central charges

$$c_L = c_R = 6Q^3, \tag{4.24}$$

and temperatures

$$T_L = \frac{(r_+ + r_-)M}{2\pi Q^3} - \frac{1}{2\pi Q}, \qquad T_R = \frac{(r_+ - r_-)M}{2\pi Q^3}. \tag{4.25}$$

For the generic nonextremal RN black hole, its hidden conformal symmetry has been discussed in [13]. For the extremal RN black hole, the scalar radial equation in the low-frequency and low momentum limit takes the form

$$\partial_r(\Delta\partial_r R(r) + \frac{2M^3(\omega-m))(2\omega-m)}{r-r_+}R(r) + \frac{M^4(\omega-m)^2}{(r-r_+)^2}R(r) = l(l+1)R, \tag{4.26}$$

where ω and m are the quantum numbers corresponding to the translation along the time and the fifth dimension. In this case, it is easy to see the equation could be rewritten as the $SL(2,R)$ Casimir (2.6) with the identification

$$\begin{aligned} &\alpha_1 = -aM, \quad \beta_1 = 2aM, \qquad \gamma_1 = M^3 a \\ &n_L = -\frac{1}{2M}, \qquad 2\pi T_L = \frac{1}{M} \end{aligned} \tag{4.27}$$

where a is just a free parameter. Once again, the left temperature is in consistency with the one found in generic case. The scattering amplitude could be computed in a similar way and is in agreement with the CFT prediction.

5. Conclusion

In this paper, we studied the hidden conformal symmetry of extremal Kerr black holes. We introduced a new set of conformal coordinates (2.1), which include five parameters $(\alpha_1, \beta_1, \gamma_1, T_L, n_L)$. It is nice to see that the induced $SL(2,R)$ quadratic Casimir is capable of rewriting the scalar Laplacian for various kinds of extremal black holes. The parameter T_L has clear physical meaning, corresponding to the temperature of the dual CFT. The other parameters take different values for various kinds of black hole and there is always one more undetermined degree of freedom among $(\alpha_1, \beta_1, \gamma_1)$. Note also the fact that all the black holes we discussed in this paper have holographic two-dimensional CFT descriptions is in accordance with the existence of hidden conformal symmetry in these black holes. We believe that this should be true for other extremal black holes with holographic pictures[40].

The microscopic descriptions of these black holes fit very nicely into their established holographic pictures, including Kerr/CFT, warped AdS/CFT, and RN/CFT et.al.. The temperatures we identified are in agreement with the ones found in the literature taking extremal limit. The Bekenstein-Hawking entropy and the scattering amplitudes are all in agreement with the CFT prediction. For the extremal black holes, their dual CFT is "chiral" in the sense that only one sector has nonvanishing temperature and the other sector is completely cooled down without excitation. This fact is already reflected in the conformal coordinates, in which only the left-moving temperature appears and the right-moving temperature is set to zero by definition.

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